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Classification of $(n-5)$-filiform Lie algebras[☆]

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Abstract

In this paper we consider the problem of classifying the $(n-5)$-filiform Lie algebras. This is the first index for which infinite parametrized families appear, as can be seen in dimension 7. Moreover we obtain large families of characteristic nilpotent Lie algebras with nilpotence index 5 and show that at least for dimension 10 there is a characteristic nilpotent Lie algebra with nilpotence index 4 which is the algebra of derivations of a nilpotent Lie algebra. © 2001 Elsevier Science Inc. All rights reserved.

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1. Generalities

Nilpotent Lie algebras have played an important role in mathematics over the last 30 years: either in the classification theory of Lie algebras, where they play a central role as a consequence of the Lévi theorem and the works of Malcev, or in the geometrical and analytical applications such as the nilmanifolds, which allow us to construct concrete compact differential manifolds, or Pfaffian systems.

The first important research about nilpotent Lie algebras is due to K. Umlauf in the last 19th century. In the 1940s and 1950s Morozov and Dixmier began with the systematical study of this class of algebras. Morozov gave a classification of

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six-dimensional nilpotent Lie algebras in 1958 [10]. The existence of an infinity of complex nilpotent Lie algebras from dimension 7 showed the complexity of the classification problem. A complete classification of seven-dimensional nilpotent Lie algebras was obtained by the first author and Goze [2].

We pointed out that for dimensions greater than or equal to 8 only partial classifications are known. Most of them correspond to the filiform Lie algebras, i.e., algebras with maximal nilpotence index. It seems natural to determine an invariant which measures the nilpotence of Lie algebras. The first author and Goze introduced in [1] an invariant that allowed not only the classification in dimension 7, but the study of nilpotent Lie algebras with lower nilpotence indexes.

Let $\mathfrak{g}_n = (\mathbb{C}^n, \mu_n)$ be a nilpotent Lie algebra. For each $X \in \mathbb{C}^n$ we denote $c(X)$ the ordered sequence of a similitude invariant of the nilpotent operator $ad_{\mathfrak{g}_n}(X)$, i.e., the ordered sequence of dimensions of the Jordan blocks for this operator. We consider the lexicographical order in the set of these sequences.

Definition 1. The characteristic sequence of $\mathfrak{g}_n$ is an *isomorphism invariant* $c(\mathfrak{g}_n)$ defined by

$$c(\mathfrak{g}_n) = \max_{X \in \mathfrak{g}_n - C^1\mathfrak{g}_n} \{c(X)\},$$

where $C^1\mathfrak{g}_n$ is the derived algebra. A nonzero vector $X \in \mathfrak{g}_n - C^1\mathfrak{g}_n$ satisfying $c(X) = c(\mathfrak{g}_n)$ is called *characteristic vector.*

Definition 2. A nilpotent Lie algebra $\mathfrak{g}_n$ is called *p-filiform* if its characteristic sequence is $c(\mathfrak{g}_n) = (n - p, 1, \ldots^{(p)}, \ldots, \ldots, 1)$.

Remark 3. It follows immediately from the definition of p-filiformness that the $(n-1)$-filiform Lie algebras are the abelian algebras $\mathfrak{a}$. It is easily shown that the $(n-2)$-filiform Lie algebras are the direct sum of a Heisenberg algebra $\mathfrak{H}_{2p+1}$ and an abelian algebra. A classification of the $(n-3)$- and $(n-4)$-filiform Lie algebras can be found in [5], respectively [3]. The former is also the last where the number of isomorphism classes is finite, as we shall see. We are primarily interested in nonsplit $(n-5)$-filiform Lie algebras, for the general $(n-5)$-filiform algebras are obtained by direct sums of nonsplit algebras and abelian algebras. Thus classifying the nonsplit we have classified all of them.

2. The classification theorem

Theorem 4 (Classification theorem). *Each n-dimensional nonsplit $(n-5)$-filiform Lie algebra $\mathfrak{g}_n$ is isomorphic to one of the laws $\mu_n^i,\ i \in \{1, \ldots, 103\}$, listed below.*

Before we give the list in even and odd dimensions we have to introduce some notations. This will be applicable to both odd and even dimensions. Let $\mathfrak{g}_n$ be an n-dimensional nilpotent complex Lie algebra. Then we identify the Lie algebra with its law $(\mathbb{C}^n, \mu_n)$, where $\mu_n \in \mathfrak{T}^a_{(2,1)}$ is an alternated tensor of type (2, 1) satisfying the Jacobi equation. We denote the derived subalgebra as $C^1\mathfrak{g}_n$. The list is structured as follows: at first we indicate indexes for which common brackets are listed. The bullet item completes the corresponding law. This kind of presentation has two advantages: on one hand side it is easier to read the concrete algebra laws, and on the other it indicates in a certain manner that the laws are closely related (as it follows in the proof). Finally, as usual the nonwritten brackets are zero or obtained by antisymmetry. For a law μ^i_n the subindex makes reference to the dimension of the algebra $\mathfrak{g}$, and the superindex to the number of the isomorphism class. Greek letters as superindexes are parameters.

2.1. Even dimension

The first subdivision is referred to as the dimension of the derived algebra.

1. $\dim C^1\mathfrak{g}_n = 6$

There are four laws with these conditions:

For the indexes $i \in \{1, 2, 3\}$ we have the common brackets

$\mu^i_{2m}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 4,$

$\mu^i_{2m}(X_5, X_2) = \mu^i_{2m}(X_3, X_4) = Y_1;\ \mu^i_{2m}(Y_{2t-1}, Y_{2t}) = X_6,\ 2 \leqslant t \leqslant m-3$ if $m > 4$.

- $\mu^1_{2m}(X_3, X_2) = Y_2;\ \mu^1_{2m}(Y_2, X_3) = X_6;\ \mu^1_{2m}(Y_2, X_2) = X_5;$
- $\mu^2_{2m}(X_3, X_2) = Y_2;$
- $\mu^3_{2m}(X_4, X_2) = X_6;\ \mu^3_{2m}(X_3, X_2) = Y_2 + X_5.$

For $i = 4$ we obtain the law

$\mu^4_{2m}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 5;$

$\mu^4_{2m}(X_5, X_2) = \mu^4_{2m}(X_3, X_4) = Y_1;\ \mu^4_{2m}(X_3, X_2) = Y_2;$

$\mu^4_{2m}(Y_3, X_3) = X_6;\ \mu^4_{2m}(Y_3, X_2) = X_5;$

$\mu^4_{2m}(Y_{2t-1}, Y_{2t}) = X_6,\ 2 \leqslant t \leqslant m-3\ .$

2. $\dim C^1\mathfrak{g}_n = 5$

For $i \in \{5, 6, 7^\alpha, 8, \ldots, 19\}$ we have

$\mu^i_{2m}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 4;$

$\mu^i_{2m}(Y_{2t-1}, Y_{2t}) = X_6,\ 2 \leqslant t \leqslant m-3$ if $m > 4$.

- $\mu^5_{2m}(X_5, X_2) = \mu^5_{2m}(X_3, X_4) = Y_1;\ \mu^5_{2m}(Y_2, X_3) = X_6;$
  $\mu^5_{2m}(Y_2, X_2) = X_5.$

- $\mu_{2m}^{6}(X_3, X_2) = Y_1$; $\mu_{2m}^{6}(Y_1, X_3) = X_6$; $\mu_{2m}^{6}(Y_1, X_2) = X_5 + X_6$; $\mu_{2m}^{6}(Y_2, X_2) = X_6$.
- $\mu_{2m}^{7,\alpha}(X_4, X_2) = \alpha X_6$; $\mu_{2m}^{7,\alpha}(X_3, X_2) = Y_1 + \alpha X_5$, $\alpha \neq 0$; $\mu_{2m}^{7,\alpha}(Y_1, X_3) = X_6$; $\mu_{2m}^{7,\alpha}(Y_1, X_2) = X_5 + X_6$; $\mu_{2m}^{7,\alpha}(Y_2, X_2) = X_6$.
- $\mu_{2m}^{8}(X_3, X_2) = Y_1$; $\mu_{2m}^{8}(Y_1, X_3) = X_6$; $\mu_{2m}^{8}(Y_1, X_2) = X_5$; $\mu_{2m}^{8}(Y_2, X_2) = X_6$.
- $\mu_{2m}^{9}(X_4, X_2) = X_6$; $\mu_{2m}^{9}(X_3, X_2) = Y_1 + X_5$; $\mu_{2m}^{9}(Y_1, X_3) = X_6$; $\mu_{2m}^{9}(Y_1, X_2) = X_5$; $\mu_{2m}^{9}(Y_2, X_2) = X_6$.
- $\mu_{2m}^{10}(X_3, X_2) = Y_1$; $\mu_{2m}^{10}(Y_1, X_2) = X_6$; $\mu_{2m}^{10}(Y_2, X_3) = X_6$; $\mu_{2m}^{10}(Y_2, X_2) = X_5$.
- $\mu_{2m}^{11}(X_5, X_2) = \mu_{2m}^{11}(X_3, X_4) = X_6$; $\mu_{2m}^{11}(X_3, X_2) = Y_1$; $\mu_{2m}^{11}(Y_1, X_2) = X_6$; $\mu_{2m}^{11}(Y_2, X_3) = X_6$; $\mu_{2m}^{11}(Y_2, X_2) = X_5$.
- $\mu_{2m}^{12}(X_3, X_2) = Y_1$; $\mu_{2m}^{12}(Y_2, X_3) = X_6$; $\mu_{2m}^{12}(Y_2, X_2) = X_5$;
- $\mu_{2m}^{13}(X_5, X_2) = \mu_{2m}^{13}(X_3, X_4) = X_6$; $\mu_{2m}^{13}(X_3, X_2) = Y_1$; $\mu_{2m}^{13}(Y_2, X_3) = X_6$; $\mu_{2m}^{13}(Y_2, X_2) = X_5$.
- $\mu_{2m}^{14}(X_5, X_2) = \mu_{2m}^{14}(X_3, X_4) = \mu_{2m}^{14}(X_4, X_2) = X_6$; $\mu_{2m}^{14}(X_3, X_2) = Y_1 + X_5$; $\mu_{2m}^{14}(Y_2, X_3) = X_6$; $\mu_{2m}^{14}(Y_2, X_2) = X_5$.
- $\mu_{2m}^{15}(X_4, X_2) = X_6$; $\mu_{2m}^{15}(X_3, X_2) = Y_1 + X_5$; $\mu_{2m}^{15}(Y_2, X_3) = X_6$; $\mu_{2m}^{15}(Y_2, X_2) = X_5$.
- $\mu_{2m}^{16}(X_3, X_2) = Y_1$; $\mu_{2m}^{16}(Y_2, X_2) = X_6$.
- $\mu_{2m}^{17}(X_5, X_2) = \mu_{2m}^{17}(X_3, X_4) = X_6$; $\mu_{2m}^{17}(X_3, X_2) = Y_1$; $\mu_{2m}^{17}(Y_2, X_2) = X_6$.
- $\mu_{2m}^{18}(X_5, X_2) = \mu_{2m}^{18}(X_3, X_4) = \mu_{2m}^{18}(X_4, X_2) = X_6$; $\mu_{2m}^{18}(X_3, X_2) = Y_1 + X_5$; $\mu_{2m}^{18}(Y_2, X_2) = X_6$.
- $\mu_{2m}^{19}(X_4, X_2) = X_6$; $\mu_{2m}^{19}(X_3, X_2) = Y_1 + X_5$; $\mu_{2m}^{19}(Y_2, X_2) = X_6$.

For $i \in \{20, 21, 22, 23\}$
$\mu_{2m}^{i}(X_1, X_j) = X_{j+1}$, $j \in \{2, 3, 4, 5\}$, $m \geqslant 5$;

$\mu_{2m}^{i}(Y_2, X_3) = X_6$; $\mu_{2m}^{i}(Y_2, X_2) = X_5$; $\mu_{2m}^{i}(Y_3, X_2) = X_6$;

$\mu_{2m}^{i}(Y_2, Y_4) = X_6$;

$\mu_{2m}^{i}(Y_{2t+1}, Y_{2t+2}) = X_6$; $2 \leqslant t \leqslant m - 4$ if $m > 5$.

- $\mu_{2m}^{20}(X_3, X_2) = Y_1$.
- $\mu_{2m}^{21}(X_5, X_2) = \mu_{2m}^{21}(X_3, X_4) = X_6;\ \mu_{2m}^{21}(X_3, X_2) = Y_1$.
- $\mu_{2m}^{22}(X_5, X_2) = \mu_{2m}^{22}(X_3, X_4) = \mu_{2m}^{22}(X_4, X_2) = X_6;$
  $\mu_{2m}^{22}(X_3, X_2) = Y_1 + X_5$.
- $\mu_{2m}^{23}(X_4, X_2) = X_6;\ \mu_{2m}^{23}(X_3, X_2) = Y_1 + X_5$.

3. $\dim C^1\mathfrak{g}_n = 4$

For the indexes $i \in \{24, \dots, 29\}$ the derived algebra $C^1\mathfrak{g}_n$ is not abelian.
For $i \in \{24, 25\}$ we have
$\mu_{2m}^{i}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 4;$

$\mu_{2m}^{i}(X_5, X_2) = \mu_{2m}^{i}(X_3, X_4) = X_6;$

$\mu_{2m}^{i}(Y_1, X_3) = X_6;\ \mu_{2m}^{i}(Y_1, X_2) = X_5;\ \mu_{2m}^{i}(Y_2, X_2) = X_6;$

$\mu_{2m}^{i}(Y_{2t-1}, Y_{2t}) = X_6,\ 2 \leqslant t \leqslant m-3$ if $m > 4$.

- $\mu_{2m}^{25}(X_4, X_2) = X_6;\ \mu_{2m}^{25}(X_3, X_2) = X_5$.

For $i \in \{26, 27\}$
$\mu_{2m}^{i}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 4;$

$\mu_{2m}^{i}(X_5, X_2) = \mu_{2m}^{i}(X_3, X_4) = X_6;$

$\mu_{2m}^{i}(Y_{2t-1}, Y_{2t}) = X_6,\ 1 \leqslant t \leqslant m-3$.

- $\mu_{2m}^{26}(Y_1, X_3) = X_6;\ \mu_{2m}^{26}(Y_1, X_2) = X_5$.
- $\mu_{2m}^{27}(X_4, X_2) = X_6;\ \mu_{2m}^{27}(X_3, X_2) = X_5;\ \mu_{2m}^{27}(Y_1, X_3) = X_6;$
  $\mu_{2m}^{27}(Y_1, X_2) = X_5$.

For $i \in \{28, 29\}$
$\mu_{2m}^{i}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 3;$

$\mu_{2m}^{i}(X_5, X_2) = \mu_{2m}^{i}(X_3, X_4) = X_6;$

$\mu_{2m}^{i}(Y_{2t-1}, Y_{2t}) = X_6,\ 1 \leqslant t \leqslant m-3$ if $m > 3$.

- $\mu_{2m}^{29}(X_4, X_2) = X_6;\ \mu_{2m}^{29}(X_3, X_2) = X_5$.

For the indexes $i \in \{30, \dots, 53\}$ the derived algebra $C^1\mathfrak{g}_n$ is abelian.
For $i \in \{30, 31, 32, 33, 34, 35, 36\}$
$\mu_{2m}^{i}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 5;$

$\mu_{2m}^{i}(Y_1, X_j) = X_{j+2},\ j \in \{2, 3, 4\};$

$\mu_{2m}^{i}(Y_2, X_j) = X_{j+3},\ j \in \{2, 3\};$

$\mu_{2m}^{i}(Y_{2t+1}, Y_{2t+2}) = X_6,\ 2 \leqslant t \leqslant m-3$ if $m > 5$.

- $\mu_{2m}^{30}(Y_3, X_2) = X_6;\ \mu_{2m}^{30}(Y_1, Y_4) = X_6.$
- $\mu_{2m}^{31}(Y_3, X_2) = X_6;\ \mu_{2m}^{31}(Y_1, Y_4) = X_6;\ \mu_{2m}^{31}(Y_2, Y_4) = X_6.$
- $\mu_{2m}^{32}(Y_3, X_2) = X_6;\ \mu_{2m}^{32}(Y_1, Y_4) = X_6;\ \mu_{2m}^{32}(Y_2, Y_3) = X_6.$
- $\mu_{2m}^{33}(Y_3, X_2) = X_6;\ \mu_{2m}^{33}(Y_1, Y_4) = X_6;\ \mu_{2m}^{33}(Y_2, Y_3) = X_6;$ $\mu_{2m}^{33}(Y_2, Y_4) = X_6.$
- $\mu_{2m}^{34}(Y_3, X_2) = X_6;\ \mu_{2m}^{34}(Y_2, Y_4) = X_6.$
- $\mu_{2m}^{35}(Y_3, X_2) = X_6;\ \mu_{2m}^{35}(Y_1, Y_3) = X_6;\ \mu_{2m}^{35}(Y_2, Y_4) = X_6.$
- $\mu_{2m}^{36}(Y_1, Y_3) = X_6;\ \mu_{2m}^{36}(Y_2, Y_4) = X_6.$

For $i \in \{37, 38, 39, 40, 41\}$
$\mu_{2m}^{i}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 4;$

$\mu_{2m}^{i}(Y_1, X_j) = X_{j+2},\ j \in \{2, 3, 4\};$

$\mu_{2m}^{i}(Y_{2t-1}, Y_{2t}) = X_6,\ 1 \leqslant t \leqslant m-3.$

- $\mu_{2m}^{37}(Y_2, X_3) = X_6;\ \mu_{2m}^{37}(Y_2, X_2) = X_5.$
- $\mu_{2m}^{38}(Y_2, X_2) = X_6.$
- $\mu_{2m}^{39}(X_3, X_2) = X_6;\ \mu_{2m}^{39}(Y_2, X_2) = X_6.$
- $\mu_{2m}^{41}(X_3, X_2) = X_6.$

For $i \in \{42, 43, 44\}$
$\mu_{2m}^{i}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 4;$

$\mu_{2m}^{i}(Y_1, X_j) = X_{j+2},\ j \in \{2, 3, 4\};$

$\mu_{2m}^{i}(Y_{2t-1}, Y_{2t}) = X_6,\ 2 \leqslant t \leqslant m-3$ if $m > 4.$

- $\mu_{2m}^{42}(Y_2, X_3) = X_6;\ \mu_{2m}^{42}(Y_2, X_2) = X_5.$
- $\mu_{2m}^{43}(Y_2, X_2) = X_6.$
- $\mu_{2m}^{44}(X_3, X_2) = X_6;\ \mu_{2m}^{44}(Y_2, X_2) = X_6.$

For $i \in \{45, 46\}$
$\mu_{2m}^{i}(X_1, X_j) = X_{j+2},\ j \in \{2, 3, 4, 5\},\ m \geqslant 4;$

$\mu_{2m}^{i}(Y_1, X_3) = X_6;\ \mu_{2m}^{i}(Y_1, X_2) = X_5;\ \mu_{2m}^{i}(Y_2, X_2) = X_6;$

$\mu_{2m}^{i}(Y_{2t-1}, Y_{2t}) = X_6,\ 2 \leqslant t \leqslant m-3$ if $m > 4.$

- $\mu_{2m}^{46}(X_4, X_2) = X_6;\ \mu_{2m}^{46}(X_3, X_2) = X_5.$

For $i \in \{47, 48, 49, 50\}$
$\mu_{2m}^{i}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 4;$

$\mu^i_{2m}(Y_1, X_3) = X_6;\ \mu^i_{2m}(Y_1, X_2) = X_5;$

$\mu^i_{2m}(Y_{2t-1}, Y_{2t}) = X_6,\ 1 \leqslant t \leqslant m-3.$

- $\mu^{47}_{2m}(Y_2, X_2) = X_6.$
- $\mu^{48}_{2m}(X_4, X_2) = X_6;\ \mu^{48}_{2m}(X_3, X_2) = X_5;\ \mu^{48}_{2m}(Y_2, X_2) = X_6.$
- $\mu^{50}_{2m}(X_4, X_2) = X_6;\ \mu^{50}_{2m}(X_3, X_2) = X_5.$

For $i \in \{51, 52, 53\}$
$\mu^i_{2m}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 3;$

$\mu^i_{2m}(Y_{2t-1}, Y_{2t}) = X_6,\ 1 \leqslant t \leqslant m-3$ if $m > 3$.

- $\mu^{52}_{2m}(X_3, X_2) = X_6.$
- $\mu^{53}_{2m}(X_4, X_2) = X_6;\ \mu^{53}_{2m}(X_3, X_2) = X_5.$

2.2. *Odd dimension*

1. $\dim C^1\mathfrak{g}_n = 6$
   There is only one law:
   $\mu^{54}_{2m+1}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 4;$

   $\mu^{54}_{2m+1}(X_5, X_2) = \mu^{54}_{2m+1}(X_3, X_4) = Y_1;\ \mu^{54}_{2m+1}(X_3, X_2) = Y_2;$

   $\mu^{54}_{2m+1}(Y_3, X_3) = X_6;\ \mu^{54}_{2m+1}(Y_3, X_2) = X_5;\ \mu^{54}_{2m+1}(Y_{2t}, Y_{2t+1}) = X_6,$
   $2 \leqslant t \leqslant m-3$ if $m > 4$.

2. $\dim C^1\mathfrak{g}_n = 5$
   For $i \in \{55, 56, 57, 58, 59, 60, 61\}$
   $\mu^i_{2m+1}(X_1, X_j) = X_{j+1},\ j \in \{2, 3, 4, 5\},\ m \geqslant 4;$

   $\mu^i_{2m+1}(Y_2, X_3) = X_6;\ \mu^i_{2m+1}(Y_2, X_2) = X_5;$

   $\mu^i_{2m+1}(Y_{2t}, Y_{2t+1}) = X_6,\ 1 \leqslant t \leqslant m-3.$

- $\mu^{55}_{2m+1}(X_5, X_2) = \mu^{55}_{2m+1}(X_3, X_4) = Y_1.$
- $\mu^{56}_{2m+1}(X_3, X_2) = Y_1;\ \mu^{56}_{2m+1}(Y_1, X_2) = X_6.$
- $\mu^{57}_{2m+1}(X_5, X_2) = \mu^{57}_{2m+1}(X_3, X_4) = X_6;\ \mu^{57}_{2m+1}(X_3, X_2) = Y_1;$
  $\mu^{57}_{2m+1}(Y_1, X_2) = X_6.$
- $\mu^{58}_{2m+1}(X_3, X_2) = Y_1.$
- $\mu^{59}_{2m+1}(X_5, X_2) = \mu^{59}_{2m+1}(X_3, X_4) = X_6;\ \mu^{59}_{2m+1}(X_3, X_2) = Y_1.$
- $\mu^{60}_{2m+1}(X_5, X_2) = \mu^{60}_{2m+1}(X_3, X_4) = \mu^{60}_{2m+1}(X_4, X_2) = X_6;$

$\mu_{2m+1}^{60}(X_3, X_2) = Y_1 + X_5$.

For $i \in \{62, \ldots, 74\}$
$\mu_{2m+1}^{i}(X_1, X_j) = X_{j+1},\ \ j \in \{2, 3, 4, 5\},\ \ m \geqslant 3;$

$\mu_{2m+1}^{i}(Y_{2t}, Y_{2t+1}) = X_6,\ \ 1 \leqslant t \leqslant m-3$ if $m > 3$.

- $\mu_{2m+1}^{62}(X_5, X_2) = \mu_{2m+1}^{62}(X_3, X_4) = Y_1;\ \ \mu_{2m+1}^{62}(X_4, X_2) = X_6;$ $\mu_{2m+1}^{62}(X_3, X_2) = X_5$.
- $\mu_{2m+1}^{63}(X_5, X_2) = \mu_{2m+1}^{63}(X_3, X_4) = Y_1;\ \ \mu_{2m+1}^{63}(X_3, X_2) = X_6$.
- $\mu_{2m+1}^{64}(X_5, X_2) = \mu_{2m+1}^{64}(X_3, X_4) = Y_1$.
- $\mu_{2m+1}^{65}(X_3, X_2) = Y_1;\ \ \mu_{2m+1}^{65}(Y_1, X_3) = X_6;\ \ \mu_{2m+1}^{65}(Y_1, X_2) = X_5 + X_6$.
- $\mu_{2m+1}^{66,\alpha}(X_4, X_2) = \alpha X_6;\ \ \mu_{2m+1}^{66,\alpha}(X_3, X_2) = Y_1 + \alpha X_5,\ \ \alpha \neq 0;$ $\mu_{2m+1}^{66,\alpha}(Y_1, X_3) = X_6;\ \mu_{2m+1}^{66,\alpha}(Y_1, X_2) = X_5 + X_6$.
- $\mu_{2m+1}^{67}(X_3, X_2) = Y_1;\ \ \mu_{2m+1}^{67}(Y_1, X_3) = X_6;\ \ \mu_{2m+1}^{67}(Y_1, X_2) = X_5$.
- $\mu_{2m+1}^{68}(X_4, X_2) = X_6;\ \ \mu_{2m+1}^{68}(X_3, X_2) = Y_1 + X_5;$ $\mu_{2m+1}^{68}(Y_1, X_3) = X_6;\ \ \mu_{2m+1}^{68}(Y_1, X_2) = X_5$.
- $\mu_{2m+1}^{69}(X_3, X_2) = Y_1;\ \ \mu_{2m+1}^{68}(Y_1, X_2) = X_6$.
- $\mu_{2m+1}^{70}(X_5, X_2) = \mu_{2m+1}^{70}(X_3, X_4) = X_6;\ \ \mu_{2m+1}^{70}(X_3, X_2) = Y_1;$ $\mu_{2m+1}^{70}(Y_1, X_2) = X_6$.
- $\mu_{2m+1}^{71}(X_3, X_2) = Y_1$.
- $\mu_{2m+1}^{72}(X_5, X_2) = \mu_{2m+1}^{72}(X_3, X_4) = X_6;\ \ \mu_{2m+1}^{72}(X_3, X_2) = Y_1$.
- $\mu_{2m+1}^{73}(X_5, X_2) = \mu_{2m+1}^{73}(X_3, X_4) = \mu_{2m+1}^{73}(X_4, X_2) = X_6;$ $\mu_{2m+1}^{73}(X_3, X_2) = Y_1 + X_5$.
- $\mu_{2m+1}^{74}(X_4, X_2) = X_6;\ \ \mu_{2m+1}^{74}(X_3, X_2) = Y_1 + X_5$.

For $i \in \{75, 76, 77, 78\}$
$\mu_{2m+1}^{i}(X_1, X_j) = X_{j+1},\ \ j \in \{2, 3, 4, 5\},\ \ m \geqslant 4;$

$\mu_{2m+1}^{i}(Y_2, X_3) = X_6;\ \ \mu_{2m+1}^{i}(Y_2, X_2) = X_5;\ \ \mu_{2m+1}^{i}(Y_3, X_2) = X_6;$

$\mu_{2m+1}^{i}(Y_{2t}, Y_{2t+1}) = X_6,\ \ 2 \leqslant t \leqslant m-3$ if $m > 4$.

- $\mu_{2m+1}^{75}(X_3, X_2) = Y_1$.
- $\mu_{2m+1}^{76}(X_5, X_2) = \mu_{2m+1}^{76}(X_3, X_4) = X_6;\ \ \mu_{2m+1}^{76}(X_3, X_2) = Y_1$.
- $\mu_{2m+1}^{77}(X_5, X_2) = \mu_{2m+1}^{77}(X_3, X_4) = \mu_{2m+1}^{77}(X_4, X_2) = X_6;$

$\mu^{77}_{2m+1}(X_3, X_2) = Y_1 + X_5.$

- $\mu^{78}_{2m+1}(X_4, X_2) = X_6;\ \mu^{78}_{2m+1}(X_3, X_2) = Y_1 + X_5.$

3. $\dim C^1\mathfrak{g}_n = 4$

For the indexes $i \in \{79, \ldots, 85\}$ the derived algebra $C^1\mathfrak{g}_n$ is not abelian.
For $i \in \{79, 80\}$

$\mu^{i}_{2m+1}(X_1, X_j) = X_{j+1},\ \ j \in \{2, 3, 4, 5\},\ \ m \geqslant 4;$

$\mu^{i}_{2m+1}(X_5, X_2) = \mu^{i}_{2m+1}(X_3, X_4) = X_6;$

$\mu^{i}_{2m+1}(Y_1, X_j) = X_{j+3},\ \ j \in \{2, 3\};$

$\mu^{i}_{2m+1}(Y_2, X_2) = \mu^{i}_{2m+1}(Y_1, Y_3) = X_6;$

$\mu^{i}_{2m+1}(Y_{2t}, Y_{2t+1}) = X_6,\ \ 2 \leqslant t \leqslant m-3 \text{ if } m > 4.$

- $\mu^{80}_{2m+1}(X_4, X_2) = X_6;\ \mu^{80}_{2m+1}(X_3, X_2) = X_5.$

For $i \in \{81, 82, 83, 84, 85\}$

$\mu^{i}_{2m+1}(X_1, X_j) = X_{j+1},\ \ j \in \{2, 3, 4, 5\},\ \ m \geqslant 3;$

$\mu^{i}_{2m+1}(X_5, X_2) = \mu^{i}_{2m+1}(X_3, X_4) = X_6;$

$\mu^{i}_{2m+1}(Y_{2t}, Y_{2t+1}) = X_6,\ \ 1 \leqslant t \leqslant m-3 \text{ if } m > 3.$

- $\mu^{81}_{2m+1}(Y_1, X_3) = X_6;\ \mu^{81}_{2m+1}(Y_1, X_2) = X_5 + X_6.$
- $\mu^{82}_{2m+1}(Y_1, X_3) = X_6;\ \mu^{82}_{2m+1}(Y_1, X_2) = X_5.$
- $\mu^{83}_{2m+1}(X_4, X_2) = X_6;\ \mu^{83}_{2m+1}(X_3, X_2) = X_5;\ \mu^{83}_{2m+1}(Y_1, X_3) = X_6;$ $\mu^{83}_{2m+1}(Y_1, X_2) = X_5.$
- $\mu^{84}_{2m+1}(Y_1, X_2) = X_6.$
- $\mu^{85}_{2m+1}(X_4, X_2) = X_6;\ \mu^{85}_{2m+1}(X_3, X_2) = X_5;\ \mu^{85}_{2m+1}(Y_1, X_2) = X_6.$

For the indexes $i \in \{86, \ldots, 103\}$ the derived algebra $C^1\mathfrak{g}_n$ is abelian.
For $i = 86$ we have

$\mu^{86}_{2m+1}(X_1, X_j) = X_{j+1},\ \ j \in \{2, 3, 4, 5\},\ \ m \geqslant 5;$

$\mu^{86}_{2m+1}(Y_1, X_j) = X_{j+2},\ \ j \in \{2, 3, 4\};$

$\mu^{86}_{2m+1}(Y_2, X_j) = X_{j+3},\ \ j \in \{2, 3\};$

$\mu^{86}_{2m+1}(Y_3, X_2) = \mu^{86}_{2m+1}(Y_1, Y_4) = X_6;$

$\mu^{86}_{2m+1}(Y_2, Y_5) = X_6;\ \ \mu^{86}_{2m+1}(Y_{2t}, Y_{2t+1}) = X_6,\ \ 3 \leqslant t \leqslant m-3 \text{ if } m > 5.$

For $i \in \{87, \ldots, 92\}$

$\mu^{i}_{2m+1}(X_1, X_j) = X_{j+1},\ \ j \in \{2, 3, 4, 5\},\ \ m \geqslant 4;$

$\mu^{i}_{2m+1}(Y_1, X_j) = X_{j+2},\ \ j \in \{2, 3, 4\};$

$\mu^{i}_{2m+1}(Y_2, X_j) = X_{j+3},\ \ j \in \{2, 3\};$

$\mu^{i}_{2m+1}(Y_{2t}, Y_{2t+1}) = X_6,\ \ 2 \leqslant t \leqslant m-3$ if $m > 4$.

- $\mu^{87}_{2m+1}(Y_3, X_2) = \mu^{87}_{2m}(Y_2, Y_3) = X_6.$
- $\mu^{88}_{2m+1}(Y_3, X_2) = X_6.$
- $\mu^{89}_{2m+1}(Y_3, X_2) = \mu^{89}_{2m+1}(Y_1, Y_2) = X_6.$
- $\mu^{90}_{2m+1}(Y_3, X_2) = \mu^{90}_{2m+1}(Y_1, Y_3) = X_6.$
- $\mu^{91}_{2m+1}(Y_1, Y_3) = X_6.$
- $\mu^{92}_{2m+1}(Y_2, Y_3) = X_6.$

For $i \in \{93, 94, 95\}$
$\mu^{i}_{2m+1}(X_1, X_j) = X_{j+1},\ \ j \in \{2, 3, 4, 5\},\ \ m \geqslant 3;$

$\mu^{i}_{2m+1}(Y_1, X_j) = X_{j+2},\ \ j \in \{3, 4\};$

$\mu^{i}_{2m+1}(Y_{2t}, Y_{2t+1}) = X_6,\ \ 1 \leqslant t \leqslant m-3$ if $m > 3$.

- $\mu^{93}_{2m+1}(Y_1, X_2) = X_4 + X_6.$
- $\mu^{94}_{2m+1}(Y_1, X_2) = X_4.$
- $\mu^{95}_{2m+1}(Y_1, X_2) = X_4;\ \ \mu^{95}_{2m+1}(X_3, X_2) = X_6.$

For $i \in \{96, 97\}$
$\mu^{i}_{2m+1}(X_1, X_j) = X_{j+1},\ \ j \in \{2, 3, 4, 5\},\ \ m \geqslant 4;$

$\mu^{i}_{2m+1}(Y_1, X_j) = X_{j+3},\ \ j \in \{2, 3\};$

$\mu^{i}_{2m+1}(Y_2, X_2) = \mu^{i}_{2m+1}(Y_1, Y_3) = X_6;$

$\mu^{i}_{2m+1}(Y_{2t}, Y_{2t+1}) = X_6,\ \ 2 \leqslant t \leqslant m-3$ if $m > 4$.

- $\mu^{97}_{2m+1}(X_4, X_2) = X_6;\ \ \mu^{97}_{2m+1}(X_3, X_2) = X_5.$

For $i \in \{98, \ldots, 103\}$
$\mu^{i}_{2m+1}(X_1, X_j) = X_{j+1},\ \ j \in \{2, 3, 4, 5\},\ \ m \geqslant 3;$

$\mu^{i}_{2m+1}(Y_{2t}, Y_{2t+1}) = X_6,\ \ 1 \leqslant t \leqslant m-3$ if $m > 3$.

- $\mu^{98}_{2m+1}(X_4, X_2) = X_6;\ \ \mu^{98}_{2m+1}(X_3, X_2) = X_5;\ \ \mu^{98}_{2m+1}(Y_1, X_3) = X_6;$
  $\mu^{98}_{2m+1}(Y_1, X_2) = X_5.$
- $\mu^{99}_{2m+1}(Y_1, X_3) = X_6;\ \ \mu^{99}_{2m+1}(Y_1, X_2) = X_5.$
- $\mu^{100}_{2m+1}(Y_1, X_3) = X_6;\ \ \mu^{100}_{2m+1}(Y_1, X_2) = X_5 + X_6.$

- $\mu_{2m+1}^{101}(Y_1, X_2) = X_6.$
- $\mu_{2m+1}^{102}(X_4, X_2) = X_6;\ \mu_{2m+1}^{102}(X_3, X_2) = X_5;\ \mu_{2m+1}^{102}(Y_1, X_2) = X_6.$
- $\mu_{2m+1}^{103}(X_3, X_2) = \mu_{2m+1}^{103}(Y_1, X_2) = X_6.$

Remark 5. As a consequence of the classification in dimension 7 (see [2]) we have $\mathfrak{g}_{2m}^{7,\alpha} \simeq \mathfrak{g}_{2m}^{7,\alpha'}$ if and only if $\alpha' = \pm\alpha$.
Similarly, $\mathfrak{g}_{2m+1}^{66,\alpha} \simeq \mathfrak{g}_{2m+1}^{66,\alpha'}$ if and only if $\alpha' = \pm\alpha$.

Comment on the proof. Let $n = \dim(\mathfrak{g})$. Then we can find a basis $(X_1, X_2, \ldots, X_6, Y_1, \ldots, Y_{n-6})$ such that X_1 is a characteristic vector and $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4,\ [X_1, X_4] = X_5,\ [X_1, X_5] = X_6,\ [X_1, X_6] = 0$ and $[X_1, Y_i] = 0\ \forall i$, which justifies the notation chosen for the basis. In the following we will use basis labelled in this manner.

From the Jacobi and nilpotence conditions, as well as the characteristic sequence it follows that the derived algebra has dimensions 6, 5 or 4. The cases are analyzed by reordering the vectors which are not in the commutator algebra in subalgebras of Heisenberg type, which is possible by elementary changes of basis. Through the application of other changes we obtain the given list. To distinguish the isomorphism classes we use classical invariants, such as the dimension of the center $Z(\mathfrak{g})$, the dimension of the Lie algebra of derivations, the weight systems, as well as the existence of concrete types of seven-dimensional ideals of characteristic sequence $(5, 1, 1)$. We additionally consider the isomorphism class of the nonsplit part of the factors of the algebras by its center, which are based on the classification of $(n-4)$-filiform Lie algebras.

As the proof is rather routine and mechanical, we omit it here. For a detailed and complete proof of the classification see: http://xxx.lanl.gov/math.RA/0012246 [4].

3. Applications

We are now interested in those obtained laws which are characteristically nilpotent, i.e., those of rank null. Characteristically nilpotent Lie algebras were first introduced by Dixmier and Lister [6], and they have become an important class of nilpotent algebras since then. Existence of such algebras has been proved for any dimension $n \geqslant 7$, as well as they do not exist for $n \leqslant 6$. There are a lot of papers constructing families of characteristically nilpotent Lie algebras (e.g., [7,8,12,13]). As most of known families and algebras are filiform, it is interesting to obtain examples of characteristically nilpotent Lie algebras which are not filiform. An interesting approach to this fact can be found in [8], where characteristically nilpotent Lie algebras are obtained from the nilradical of Borel subalgebras of complex simple Lie algebras. We will consider here the p-filiform, characteristically nilpotent Lie algebras.

Definition 6. A Lie algebra $\mathfrak{g}_n$ is called characteristically nilpotent if each derivation $f \in \mathrm{Der}(\mathfrak{g}_n)$ is nilpotent.

This definition results from a generalization of the descending central sequence given by Dixmier and Lister. Unfortunately very little is known about the algebra of derivations of a nilpotent Lie algebra, so that a direct construction of a nilpotent Lie algebra of derivations is not a trivial problem [6,13]. However, characteristically nilpotent Lie algebras behave as desired with sums, i.e., an algebra that is a finite sum of ideals is characteristically nilpotent if and only if each ideal is characteristically nilpotent [9]. From this we see that $(n-p)$-filiform characteristically nilpotent Lie algebras must be searched among the nonsplit ones.

Proposition 7. *For $p \leqslant 4$ there do not exist $(n-p)$-filiform characteristically nilpotent Lie algebras.*

Proof. For the abelian and the Heisenberg algebras the assertion is evident. For $p=3$ and 4 the proposition follows from the fact that these algebras have all ranks greater than or equal to one (see [3,5]). □

Remark 8. It follows that characteristically nilpotent Lie algebras whose nilindex is four must have characteristic sequence $\geqslant (4, 2, \ldots, 1)$.

Proposition 9. *An $(n-5)$-filiform Lie algebra is characteristically nilpotent if and only if it is isomorphic to one of the following laws*: $\mathfrak{g}_7^{65}$, $\mathfrak{g}_7^{66,\alpha}$ $(\alpha \neq 0)$, $\mathfrak{g}_7^{68}$, $\mathfrak{g}_7^{70}$, $\mathfrak{g}_7^{81}$, $\mathfrak{g}_7^{83}$, $\mathfrak{g}_8^{6}$, $\mathfrak{g}_8^{7,\alpha}$ $(\alpha \neq 0)$, $\mathfrak{g}_8^{9}$, $\mathfrak{g}_8^{11}$, $\mathfrak{g}_8^{25}$, $\mathfrak{g}_8^{27}$, $\mathfrak{g}_9^{57}$, $\mathfrak{g}_9^{80}$.

Corollary 10. *There are characteristic nilpotent Lie algebras $\mathfrak{g}_n$ with nilpotence index* 5 *for $n = 7, 8, 9, 14, 15, 16, 17, 18$ and $n \geqslant 21$.*

We have seen that the Lie algebra of derivations of a nilpotent Lie algebra does not have to be nilpotent in general. In fact the possibilities for the algebra of derivations of a nilpotent Lie algebra are very ample. They can vary from representations of the special linear algebra $\mathfrak{sl}_n$ to nilpotent Lie algebras, and no guide has been recognized until now. So it is natural to ask for the existence of characteristically nilpotent Lie algebras whose algebra of derivations has concrete properties: specifically we ask if there are characteristically nilpotent Lie algebras of derivations. That this does not always occur is shown by the following example:

Example 11. Let $\mathfrak{g}_8^6$ be the Lie algebra whose law is μ_8^6. The algebra of derivations has dimension 13 and is isomorphic to

$$\begin{array}{lll} [Z_1, Z_2] = Z_3, & [Z_2, Z_3] = -Z_6, & [Z_3, Z_{10}] = -Z_5 \\ [Z_1, Z_3] = Z_4, & [Z_2, Z_6] = -Z_5, & [Z_3, Z_{13}] = -Z_5 \\ [Z_1, Z_4] = Z_5, & [Z_2, Z_9] = -Z_6, & [Z_8, Z_{11}] = -Z_5 \\ [Z_1, Z_{10}] = -Z_6, & [Z_2, Z_{10}] = -Z_6, & [Z_8, Z_{12}] = Z_7 \\ [Z_1, Z_{11}] = -Z_7, & [Z_2, Z_{12}] = Z_5, & [Z_9, Z_{10}] = Z_5 \\ & [Z_2, Z_{13}] = -Z_4 & \end{array}$$

The linear system (S) associated to this algebra has the nontrivial solution

$$v = (\lambda_i)_{1 \leqslant i \leqslant 13} = \lambda(1, 1, 2, 3, 4, 3, 4, 1, 2, 2, 3, 3, 2),$$

so this algebra has nontrivial rank.

It seems that almost all characteristically nilpotent Lie algebras will have a non characteristically nilpotent Lie algebra of derivations.The existence of algebras with characteristically nilpotent algebra of derivations is proven by the next example, which gives a positive answer to the question formulated by Tôgô in [11]:

Example 12. For the algebra $\mathfrak{g}_7^{81}$ with associated law μ_7^{81} the algebra of derivations $Der(\mathfrak{g}_7^{81})$ has dimension 10 and is isomorphic to the following algebra:

$$\begin{array}{lll} [Z_1, Z_2] = Z_3, & [Z_2, Z_6] = -Z_5, & [Z_7, Z_8] = 2Z_5 - 2Z_6 + 2Z_{10} \\ [Z_1, Z_3] = Z_4, & [Z_2, Z_8] = -Z_6, & [Z_7, Z_9] = Z_5 - 2Z_6 + 2Z_{10} \\ [Z_1, Z_4] = Z_5, & [Z_2, Z_9] = -Z_4 - 2Z_6, & [Z_8, Z_9] = 2Z_6 - 2Z_{10} \\ [Z_1, Z_7] = -Z_4, & [Z_2, Z_{10}] = -Z_5, & \\ [Z_1, Z_8] = -Z_6, & [Z_3, Z_8] = -Z_5, & \\ & [Z_3, Z_9] = -Z_5, & \end{array}$$

It is not difficult to prove that this algebra is characteristically nilpotent.

Remark 13. Thus it is possible to define an “index” for characteristically nilpotent Lie algebras. The index equal to 1 corresponds to the characteristically nilpotent algebras like $\mathfrak{g}_8^6$, i.e., those whose algebra of derivations admits a nontrivial diagonalizable derivation. So we can call a Lie algebra $\mathfrak{g}$ characteristically nilpotent of index k if $\mathfrak{g}$ and the $(k-1)$ first algebras of derivations are characteristically nilpotent and the k^{th} algebra of derivations is not characteristically nilpotent. It would be interesting to know if there is a relation between the nilpotence index or the characteristic sequence of the algebra and the index k defined above. It would also be interesting to know if the sequence of derivation algebras stabilizes or not.

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